W7/2 Primary Decomposition

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Thm 5.5 (1st Uniqueness Theorem) If $I = \bigcap_{i=1}^n Q_i$ is an irredundant primary decomposition, then $\sqrt{Q_1}, \dots, \sqrt{Q_n}$ are precisely the prime ideals of the form $\sqrt{(I:a)}$, $a \in A$.

In particular: n and $\sqrt{Q_1}, \dots, \sqrt{Q_n}$ are independent of the particular decomposition.

Proof: For $a \in A$: $\quad (I:a) = \left(\bigcap_{i=1}^n Q_i : a\right) = \bigcap_{i=1}^n (Q_i : a)$

$$\Rightarrow \sqrt{(I:a)} \overset{C2.4}{=} \bigcap_{i=1}^n \sqrt{(Q_i:a)} \overset{L6.4}{=} \bigcap_{\substack{i=1 \\ a \notin Q_i}}^n \sqrt{Q_i}. \quad (*)$$

- If $\sqrt{(I:a)} = P$ is prime, then $P = \sqrt{Q_{i_1}} \cap \dots \cap \sqrt{Q_{i_\ell}} \Rightarrow P \subseteq \sqrt{Q_{i_j}} \overset{P \text{ prime}}{\subseteq} P$ for some j, hence $P = \sqrt{Q_{i_j}}$.
- Let $1 \leq i \leq n$ and $a \in \bigcap_{\substack{j=1 \\ j \neq i}}^n Q_j \setminus Q_i \overset{(*)}{\Rightarrow} \sqrt{(I:a)} = \sqrt{Q_i}$.

$\square$

Def: The primes $\sqrt{Q_1}, \dots, \sqrt{Q_n}$ are the associated primes of I (= primes belonging to I)

$\sqrt{Q_i}$ is an isolated (= minimal) prime of I if it is minimal in $\{\sqrt{Q_1}, \dots, \sqrt{Q_n}\}$, embedded otherwise.

Exm: $I = (x^2, xy) = (x) \cap (x,y)^2 = (x) \cap (x^2, y)$

assoc. primes: $\underset{\text{isolated}}{(x)}$, $\underset{\text{embedded}}{(x,y)}$

Prop 5.6 If $I \trianglelefteq A$ has a primary decomposition, then $\{\text{isolated primes assoc. to } I\} = \{\text{min. primes over } I\}$.

Proof: „$\supseteq$": Let $P \in \operatorname{Spec}(A)$, $P \supseteq I = \bigcap_{i=1}^n Q_i \Rightarrow P \supseteq \sqrt{Q_1} \cap \dots \cap \sqrt{Q_n}$

Proof: „$\supseteq$": Let $P \in \mathrm{Spec}(A)$, $P \supseteq I = \bigcap_{i=1}^{n} Q_i \Rightarrow P \supseteq \sqrt{Q_1} \cap \ldots \cap \sqrt{Q_n}$

$\Rightarrow P \supseteq \bigcap_{\substack{i \\ \sqrt{Q_i} \text{ isolated}}} \sqrt{Q_i} \Rightarrow \exists$ isolated $\sqrt{Q_i}$ with $P \supseteq \sqrt{Q_i}$

If P is minimal over I, $P = \sqrt{Q_i}$.

„$\subseteq$" Let P be an isolated prime of I. Then $P \supseteq I$. If $P \supseteq Q \supseteq I$ for some prime Q, then by „$\supseteq$" there is an i s.t. $\sqrt{Q_i} \subseteq Q \subseteq P \underset{P \text{ isolated}}{\Rightarrow} \sqrt{Q_i} = Q = P$. $\square$

Prop 5.7 If $\underline{0} = \bigcap_{i=1}^{n} Q_i$ is an irredundant primary decomposition, $P_i := \sqrt{Q_i}$, then
$$\bigcup_{i=1}^{n} P_i = \{\text{zero-divisors of } A\}$$

Proof: Let $Z \subseteq A$ be the set of zero-divisors.

$b \in Z \Leftrightarrow \exists a \in A \setminus \{0\}: ba = 0 \Leftrightarrow \exists a \in A \setminus \{0\}: b \in (\underline{0} : a)$

Proof of T5.5

$$Z = \bigcup_{0 \neq a \in A} (\underline{0} : a) = \bigcup_{0 \neq a \in A} \sqrt{(\underline{0} : a)} = \bigcup_{0 \neq a \in A} \bigcap_{a \notin Q_j} P_j \subseteq \bigcup_{i=1}^{n} P_i.$$

Conversely, by T5.5, for each P_i, there is an $a \in A \setminus \{0\}$ s.t. $P_i = \sqrt{(\underline{0} : a)}$ $\square$

Lemma 5.8 Let $S \subseteq A$ be an m.c. set, $j: A \to S^{-1}A$ localization

There is a bijection

$$\{Q \subsetneq A : Q \text{ primary}, Q \cap S = \emptyset\} \longleftrightarrow \{\text{primary ideals of } S^{-1}A\}$$
$$Q \longmapsto S^{-1}Q$$
$$j^{-1}(Q) \longleftarrow Q$$

Proof: If $Q \trianglelefteq S^{-1}A$ is primary, so is $j^{-1}(Q)$, and $S^{-1}j^{-1}(Q) = Q$.

Let $Q \subsetneq A$ be primary, $S \cap Q = \emptyset$. Note: $S \cap Q = \emptyset \Leftrightarrow S \cap \sqrt{Q} = \emptyset$.

Suppose $\frac{a}{s}, \frac{b}{t} \in S^{-1}A$ are s.t. $\frac{ab}{st} = \frac{q}{u}$ with $q \in Q$, $u \in S$.

$\Rightarrow \exists v \in S: \underbrace{uv}_{\in S}ab = qstv \in Q \overset{S \cap \sqrt{Q} = \emptyset}{\Longrightarrow} uv \notin \sqrt{Q}$, so $ab \in Q \Rightarrow a \in Q$ or $b \in \sqrt{Q}$

$\Rightarrow \frac{a}{s} \in S^{-1}Q$ or $\frac{b}{t} \in \sqrt{S^{-1}Q}$

$\underset{\in S}{\bigcup}$

$\Rightarrow \frac{a}{s} \in S^{-1}Q$ or $\frac{b}{t} \in \sqrt{S^{-1}Q}$.

So $S^{-1}Q$ is primary.

$$j^{-1}(S^{-1}Q) \overset{P2.1}{=} \bigcup_{s\in S} (Q:s) \overset{S\cap\sqrt{Q}=\emptyset,\ L5.4}{=} Q \qquad \square$$

<u>Note:</u> Since $S^{-1}\sqrt{Q} = \sqrt{S^{-1}Q}$ (HW 1, Problem 5.2), if Q is P-primary, $S^{-1}Q$ is $S^{-1}P$-primary.

A set X of primes belonging to $I \trianglelefteq A$ is isolated if whenever $P \in X$ and $P' \subseteq P$ is another prime belonging to I, also $P' \in X$.

<u>Thm 5.9</u> (2nd Uniqueness Thm) Suppose $I \trianglelefteq A$ has an irredundant primary decomposition $I = \bigcap_{i=1}^{n} Q_i$, and $P_i = \sqrt{Q_i}$. If X is an isolated set of primes of I, then $\bigcap_{P_i \in X} Q_i$ is uniquely determined by I.

<u>In particular:</u> Q_i for P_i isolated is unique.

<u>Proof:</u> Let $S = A \setminus \bigcup_{P\in X} P$. Thus $P_i \in X \Leftrightarrow P_i \cap S = \emptyset \Leftrightarrow Q_i \cap S = \emptyset$.

So
$$S^{-1}I = \bigcap_{\substack{i=1 \\ P_i \in X}}^{n} S^{-1}Q_i$$

If $j: A \to S^{-1}A$ is the localisation, then

$$\hat{I} := j^{-1}(S^{-1}I) = \bigcap_{\substack{i=1 \\ P_i \in X}}^{n} Q_i \text{ is determined by } I \text{ and } X. \qquad \square$$